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ADVANCES IN THE NUMERICAL COMPUTATION OF CAPILLARY-GRAVITY WAVES

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§1. INTRODUCTION

This paper deals with the computation of symmetric finite amplitude waves propagating without change of form on the surface of a liquid above a horizontal flat bottom. We assume the liquid to be inviscid and incompressible and the flow to be irrotational. The free-surface condition, including the effects of capillarity, is used in its exact nonlinear form.

In Sec. 2 we formulate the problem as an integro-differential equation system for the unknown shape of the free surface. This system consists of a nonlinear differential equation coupled with a linear integral equation. A numerical scheme based on Newton's iterations is derived to solve these equations. Details of the numerical procedure are given in Sec. 3. The formulation of the problem and the numerical method used to solve it, follows closely the work of Schwartz and Vanden-Broeck [8] and Vanden-Broeck and Schwartz [10].

In recent years important progress has been achieved in the calculation of steep gravity waves in water of arbitrary uniform depth. For example, Schwartz [7] extended Stokes' series to high order by computer methods and then recast these polynomials as Padé approximants. High accuracy solutions were obtained in that way. This approach was further refined by Longuet-Higgins [4] in the infinite depth case and by Cokelet [2] in the finite depth case. In Sec. 4.1 we use the

numerical procedure of Secs. 2 and 3 to compute steep gravity waves in shallow water. These computations numerically confirm the validity of the use of Padé approximants as applied to gravity waves. While the convergence of Cokelet's series deteriorates for steep waves in very shallow water, our numerical scheme remains efficient for depths as small as 1/120 of a wavelength.

Approximate solutions for gravity-capillary waves were published long ago by Harrison [3]. The surface profile was sought as a Fourier series in the horizontal coordinate with coefficients that are power series in the wave amplitude. This perturbation expansion invoked Stokes' hypothesis that the n-th Fourier coefficient is of n-th order in the amplitude. The use of this hypothesis resulted in infinite values for certain of the series coefficients when the dimensionless capillary number κ is the reciprocal of an integer greater than one. For the particular case $\kappa = 1/2$, Wilton [11], by revoking Stokes' hypothesis and reordering the terms of the series, was able to find two solutions. This analytical approach was further extended by Pierson and Fife [6], Nayfeh [5], and Chen and Saffman [1]. The numerical scheme of Sec. 3 was applied by Schwartz and Vanden-Broeck [8] to compute gravity-capillary waves in deep water. A summary of their results is given in Sec. 4.2. In addition, we show that the wave speed parameter μ is not a single valued function of κ. This fact enables us to extend Schwartz and Vanden-Broeck's results.

All the capillary-gravity waves obtained by Schwartz and Vanden-Broeck [8] are ultimately limited by contact with adjacent waves. Vanden-Broeck and Keller [9] have developed a numerical procedure to construct waves of higher amplitude. A discussion of their method is given in Sec. 4.3.

§2. MATHEMATICAL FORMULATION

We consider two-dimensional, periodic waves of wavelength λ and phase velocity c propagating on the surface of a liquid under the combined effects of gravity g and surface tension T over a horizontal bottom. We choose a frame of reference in

which the waves are steady, as is the fluid motion, which is assumed to be a potential flow.

The variables are made dimensionless by referring them to the velocity scale $(g\lambda/2\pi)^{1/2}$ and to the length $\lambda/2\pi$. In addition, we introduce the wave-speed parameter

$$\mu = \frac{2\pi c^2}{g\lambda} \tag{1}$$

and the dimensionless capillary number

$$\kappa = \frac{4\pi^2 T}{\rho g \lambda^2} \tag{2}$$

Here ρ is the fluid density.

The condition of constant pressure ($p = 0$) on the free surface can be written

$$\frac{1}{2} qq^* + y + \frac{\kappa}{R} = \frac{\mu}{2} \tag{3}$$

Here, $q = u - iv$ is the complex velocity and the asterisk signifies complex conjugation. The acceleration of gravity acts in the negative y direction. The y-axis is chosen as a line of symmetry of a surface wave. R is the radius of curvature counted positive when the center of curvature lies on the fluid side of the free surface.

The choice of the Bernoulli constant in (3) fixes the origin of y as the undisturbed level of the free surface for which the velocity and the curvature are, respectively, equal to $\mu^{1/2}$ and zero.

Let the stream function assume the values zero and $-Q$ on the free surface and on the bottom, respectively. The undisturbed fluid depth d is defined by

$$d = \frac{Q}{\sqrt{\mu}} \tag{4}$$

We choose the complex potential

$$f = \phi + i\psi$$

as the independent variable.

In order to satisfy the boundary condition $\partial y/\partial \phi = 0$ on the bottom $\psi = -Q$, we reflect the flow in the boundary $\psi = -Q$.

Thus we seek $z = x + iy$ as an analytic function of f in the strip $-2Q \le \psi \le 0$.

It is convenient to introduce the following change of variables

$$f = \phi + i\psi = i\sqrt{\mu}\,\log \zeta \tag{5}$$

where $\zeta = re^{i\theta}$. Relation (5) maps the bottom $\psi = -Q$, the free surface $\psi = 0$, and its image $\psi = -2Q$, respectively, onto the circles $r = r_0 = e^{-d}$, $r = 1$, and $r = r_0^2$. The values of the real and imaginary parts of the function $F(\zeta) = \zeta dz/d\zeta - i$ on the free surface $r = 1$ and on its image $r = r_0^2$ are related by the identities

$$x'(\theta) = -1 - \mathcal{I}F(e^{i\theta}) = -1 - \mathcal{I}F(r_0^2 e^{i\theta}) \tag{6}$$

$$y'(\theta) = \mathcal{R}F(e^{i\theta}) = -\mathcal{R}F(r_0^2 e^{i\theta}) \tag{7}$$

Here, $x'(\theta)$ and $y'(\theta)$ represent, respectively, the real and imaginary parts of the function $dz/d\theta$ on the free surface $r = 1$. These variables have the periodicity property

$$\int_{-\pi}^{+\pi} [x'(\theta) + 1]\, d\theta = \int_{-\pi}^{+\pi} y'(\theta)\, d\theta = 0 \tag{8}$$

In order to find a relation between $x'(\theta)$ and $y'(\theta)$ we apply Cauchy's theorem to the function $F(\zeta)$ in the annulus $r_0^2 \le |\zeta| \le 1$. Using the relations (6), (7), and (8) we find, after some algebra,

$$x'(\theta) + 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y'(\phi)\cot\frac{(\phi - \theta)}{2}\, d\phi$$
$$- \frac{r_0^2}{\pi}\int_{-\pi}^{\pi} \frac{[1 + x'(\theta)][r_0^2 - \cos(\theta - \phi)] - y'(\phi)\sin(\phi - \theta)}{r_0^4 - 2r_0^2\cos(\theta - \phi) + 1}\, d\phi \tag{9}$$

the first integral being of Cauchy principal value form. Exploiting the bilateral symmetry of the wave about $\theta = 0$, we rewrite (9) as

$$x'(\theta) + 1 = \frac{1}{2\pi}\int_0^\pi y'(\phi)(\cot \tfrac{1}{2}(\phi - \theta) + \cot \tfrac{1}{2}(\phi + \theta))d\phi$$

$$- \frac{r_0^2}{\pi}\int_0^\pi \frac{[1 + x'(\phi)][r_0^2 - \cos(\theta - \phi)] - y'(\phi)\sin(\phi - \theta)}{r_0^4 - 2r_0^2 \cos(\theta - \phi) + 1} d\phi$$

$$- \frac{r_0^2}{\pi}\int_0^\pi \frac{[1 + x'(\phi)][r_0^2 - \cos(\theta + \phi)] - y'(\phi)\sin(\phi + \theta)}{r_0^4 - 2r_0^2 \cos(\theta + \phi) + 1} d\phi \qquad (10)$$

The surface condition (3) can now be rewritten as

$$y + \frac{\mu}{2}[(x'^2 + y'^2)^{-1} - 1]$$
$$+ \kappa(x'y'' - y'x'')(x'^2 + y'^2)^{-3/2} = 0 \qquad (11)$$

In addition to the parameters r_0, κ, and μ, a given wave is characterized by a third parameter s, which is a measure of the wave steepness. This parameter can be defined in many different ways. Precise definitions appropriate to each of the problems considered will be given in the following sections. *Dimensional analysis implies that a functional relationship* should exist among these parameters:

$$f(\mu,\kappa,s,r_0) = 0$$

We shall consider two closely related numerical schemes. In the numerical scheme I, we fix r_0, s, and κ and we seek the function $y(\theta)$, $\theta \in [0,\pi]$ and a value for μ. In the numerical scheme II, we fix r_0, s, and μ and we seek the function $y(\theta)$ and a value for κ.

§3. NUMERICAL PROCEDURE

We seek a numerical solution of the integro-differential system of Eqs. (10) and (11) by a finite difference method. Introducing a uniform mesh, we have

$$\theta_i = [(i - 1)/N]\pi \qquad i = 1, \ldots, N + 1 \qquad (12)$$

Since the wave is symmetrical, we have

$$\left(\frac{\partial y}{\partial \theta}\right)_{\theta=\theta_1} = \left(\frac{\partial y}{\partial \theta}\right)_{\theta=\theta_{N+1}} = 0$$

Thus the unknown function $\partial y/\partial\theta$ can be represented by the vector of dimension N - 1,

$$\underline{y}' = (y_{\theta_2}, \ldots, y_{\theta_N}) \tag{13}$$

where

$$y_{\theta_i} = \left(\frac{\partial y}{\partial \theta}\right)_{\theta=\theta_i}$$

We now define the midpoints $\gamma_i = \frac{1}{2}(\theta_i + \theta_{i+1})$, $i = 1, \ldots, N$, and we represent the values of $\partial x/\partial\theta$, $\partial y/\partial\theta$, and y at the points γ_i by the vectors $\underline{x}_M'$, $\underline{y}_M'$, and $\underline{y}_M$.

We seek to satisfy the system of Eqs. (10) and (11) at the N points γ_i. The integrals in (10) are evaluated at the points γ_i by the trapezoidal rule (which is of infinite order since the integrand is periodic). The integrals involving the function $y'(\phi)$ are computed by integration over θ_i. The singularity of the Cauchy principal value is automatically taken into account since the quadrature is symmetrical with respect to the singularity. The remaining integrals in (10) are computed by integration over γ_i. Thus, we obtain

$$\underline{1} + \underline{x}_M' = A\underline{y}' + B(\underline{x}_M' + \underline{1})$$

or

$$x_M' = -\underline{1} + (I - B)^{-1}A\underline{y}' \tag{14}$$

Here, A and B are known matrices and I is the unit matrix. We note that B = 0 for infinite depth (i.e., $r_0 = 0$) so that no matrix inversion is needed in this particular case.

The vectors $\underline{y}_M'$ and $\underline{y}_M$ are expressed in terms of $\underline{y}'$ by a sixth-order interpolation formula and by a sixth-order quadrature formula, thus,

$$\underline{y}_M' = C\underline{y}' \tag{15}$$

$$\underline{y}_M = y_0 + D\underline{y}' \tag{16}$$

where C and D are known matrices. The elevation y_0 of the free surface at $\theta = 0$ has to be found as part of the solution.

Substituting (14), (15), and (16) into (11), we obtain a system of N nonlinear algebraic equations. Using the definition of s we obtain an extra equation. Thus we have N + 1 equations for the N + 1 unknowns (y_2', ..., y_N', y_0, μ) in scheme I and (y_2', ..., y_N', y_0, κ) in scheme II. This system of equations was solved by Newton's iterations. Each iteration requires the inversion of a matrix. We note that the matrices A and B in (14) depend only on r_0. Thus computing time can be saved by computing $(I - B)^{-1}A$ at the beginning of the program.

In all the cases to be discussed, the numerical scheme was found to converge quadratically. Four or five iterations were required tc satisfy the algebraic equations with an error less than 10^{-12}. With N = 80, each iteration takes about 3.5 seconds on a CDC 6600 computer.

§4. DISCUSSION OF RESULTS

§4.1 Gravity Waves in Shallow Water

Vanden-Broeck and Schwartz [10] have applied the numerical scheme of Sec. 3 to compute steep gravity waves in shallow water. The effect of surface tension was neglected (i.e., $\kappa = 0$) and the parameter s was chosen to be

$$s = \frac{2y_c}{\mu} \tag{17}$$

Here, y_c is the elevation of the crests of the wave. For the highest wave, the velocity at the crest vanishes. Thus, from (3), $y_c = \mu/2$, so s = 1 for the highest wave. In general s ranges between 0 and 1.

In order to compute accurately the crests of the waves, which become sharp when s increases, a new independent variable β was introduced by the relation

$$\theta = \beta - \alpha \sin \beta \tag{18}$$

The closer α is to one, the greater the concentration of mesh points near the crest. It was found that steep waves could be computed by choosing $\alpha = 0.999$.

Table 1 shows values of μ for $r_0 = 0.5$ computed with 40, 60, and 80 mesh points. The computed values for N = 80 have

TABLE 1. Values of μ for $r_0 = 0.5$ and $0.53 \le s \le 0.992$.

s	N = 40	N = 60	N = 80	Cokelet
0.52966	0.666501	0.666501	0.666501	0.666501
0.69331	0.706443	0.706443	0.706443	0.706443
0.84727	0.748230	0.748230	0.748230	0.748230
0.92326	0.764404	0.764403	0.764403	0.764403
0.96149	0.767749	0.767750	0.767750	0.767748
0.97687	0.767537	0.767540	0.767540	0.76754
0.98458	0.767089	0.767096	0.767097	0.76707
0.99228	0.766551	0.766556	0.766557	0.76648

converged to six places following the decimal point for $s < 0.97$ and to five places for $s > 0.97$. The last column contains the values of μ obtained by Cokelet [2]. The lowest number corresponds to the highest steepness for which Cokelet computed the wave speed. His results are in good agreement with our numerical values. Thus, the validity of the Padé approximant method for finite depth gravity waves is confirmed. It should be pointed out that Cokelet used another parameter instead of our parameter s. Thus, a part of the small discrepancy at the bottom of the table may be attributed to the loss of accuracy involved in the calculation of s from Cokelet's results.

The convergence of Cokelet's results deteriorates for steep waves in shallow depth. On the other hand, our numerical scheme remains efficient for depths as small as 1/120 of a

TABLE 2. Values of μ for $r_0 = 0.9$.

s	N = 60	N = 80	Cokelet
0.42183	0.127329	0.127329	0.12733
0.61793	0.142055	0.142054	0.141983
0.81142	0.158013	0.158011	0.1578
0.90849	0.164092	0.164091	0.1638
0.95	0.165039	0.165038	0.164
0.98	0.164678	0.164684	0.163
0.99	0.164433	0.164437	0.162

wavelength. In Table 2 we compare the two methods for the smallest depth $r_0 = 0.9$ considered by Cokelet. For steep waves our results have converged to five decimal places. Cokelet's series expansion method gives only two correct decimal places for steep waves.

§4.2 Gravity-Capillary Waves in Deep Water

The procedure of Sec. 3 was used by Schwartz and Vanden-Broeck [8] to compute capillary-gravity waves in deep water. The parameter s was defined as the steepness of the wave (i.e., the difference of ordinates between a crest and a trough divided by the wavelength). Four families, each labeled with a "type number," were studied. The numerical results indicate that Harrison's [3] series solution agrees, at least for small finite values of the steepness, with the family of type n when κ lies in the open interval $(1/n + 1, 1/n)$ for $n \geq 2$ and in the interval $(1/2, \infty)$ for $n = 1$. Typical profiles of the free surface for various values of μ, κ, and s can be found in Schwartz and Vanden-Broeck [8].

Figure 1 shows the variation of the parameter μ with κ for $s = 0.03$. The four continuous families are displayed. Also shown is the well-known infinitesimal wave solution

$$\mu = 1 + \kappa$$

Schwartz and Vanden-Broeck [8] started the iterations with a simple cosine profile. The numerical scheme was found to converge to the family of type n when $\kappa \in (1/n + 1, 1/n)$ for $n \geq 2$ and $\kappa \in (1/2, \infty)$ for $n = 1$. Once a given wave of type n had been obtained, the family n was then computed by a type of "boot-strap" technique, using the numerical scheme I. That is, a converged solution for one value of κ was used as an initial guess for a wave with κ altered by a few percent, and so on. For $s = 0.03$ Schwartz and Vanden-Broeck [8] found that family 3 could not be computed for $\kappa \simeq 0.2515$. Similarly, family 4 could only be computed for $\kappa \geq 0.21$.

In the present paper we have extended the computations of families 3 and 4 by a "boot-strap" technique using the numerical scheme II. For example the solution of family 3

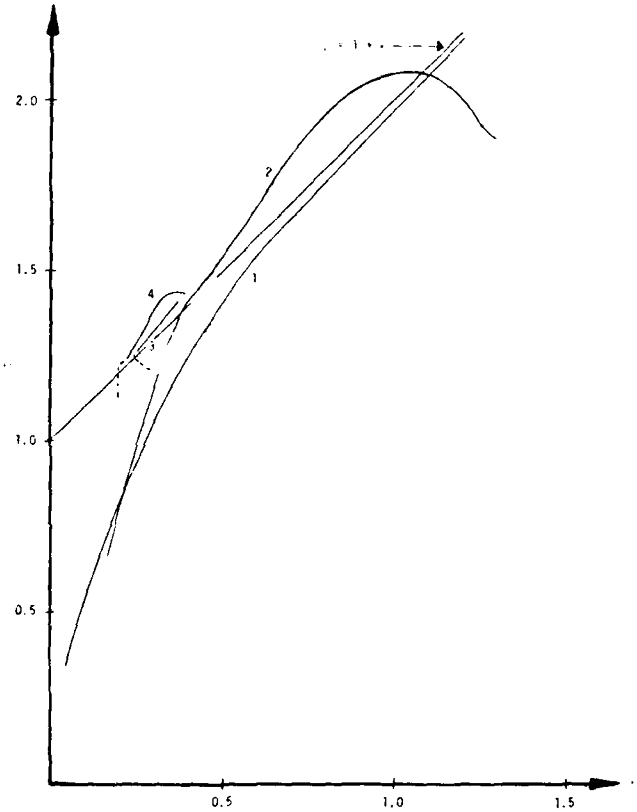

FIG. 1. Variation of speed parameter μ with capillary number κ for $s = 0.03$. The solid curves are results of Schwartz and Vanden-Broeck [8] while the dashed lines show the results of the present computations.

for $\kappa = 0.2515$ and $\mu = 1.2478$ was used as an initial guess to compute the solution for a value of μ slightly less and so on. The results are shown in Fig. 1. Families 3 and 4 could be extended to larger ranges than those presented in the figure. However, these results are not reported here since our purpose is simply to show that the parameter μ is not a single valued function of κ.

§4.3 Waves with Trapped Bubbles

All the capillary-gravity waves computed by Schwartz and Vanden-Broeck [8] are ultimately limited by contact with adjacent waves for some value s* of the amplitude parameter. The analytic continuation of these solutions for s > s* yield overlapping waves with multiple valued velocities. Thus, these solutions are not admissible as solutions of the physical problem. Vanden-Broeck and Keller [9] have shown how to modify the numerical scheme of Secs. 2 and 3 in order to obtain physically acceptable solutions for s > s*. The general idea is to prevent the free surface from crossing itself. The solutions for s > s* have adjacent waves touching at one point, just like in Schwartz and Vanden-Broeck's highest waves. Thus each pair of adjacent waves enclose a region devoid of fluid, which we call a bubble. The pressure in the bubble is a function of the amplitude parameter.

Vanden-Broeck and Keller [9] have performed accurate computations in the particular case of pure capillary waves in water of infinite depth. They have shown that the new family of solutions exists for arbitrarily large values of the steepness.

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REFERENCES

1. Chen, B., and Saffman, P. G., Steady gravity-capillary waves on deep water, Part I: Weakly nonlinear waves, Stud. Appl. Math. 60:183-210 (1979).

2. Cokelet, E. D., Steep gravity waves in water of arbitrary uniform depth, Phil. Trans. Roy. Soc. A 286:183-230 (1977).

3. Harrison, W. J., The influence of viscosity and capillarity on waves of finite amplitude, Proc. Lond. Math. Soc. 7:107-121 (1909).

4. Longuet-Higgins, M. S., Integral properties of periodic gravity waves of finite amplitude, Proc. Roy. Soc. A 342:157-174 (1975).

5. Nayfeh, A. H., Triple and quintuple-dimpled wave profiles in deep water, Phys. Fluids 13:545-550 (1970).

6. Pierson, W. J., and Fife, P., Some nonlinear properties of long crested periodic waves with lengths near 2.44 centimeters, J. Geophys. Res. 66:163-179 (1961).

7. Schwartz, L. W., Computer extension and analytic continuation of Stokes' expansion for gravity waves, J. Fluid Mech. 62:553-578 (1974).

8. Schwartz, L. W., and Vanden-Broeck, J.-M., Numerical solution of the exact equations for capillary-gravity waves, J. Fluid Mech. 95:119-139 (1979).

9. Vanden-Broeck, J.-M., and Keller, J. B., A new family of capillary waves, J. Fluid Mech. (in press).

10. Vanden-Broeck, J.-M., and Schwartz, L. W., Numerical computation of steep gravity waves in shallow water, Phys. Fluids 22:1868-1871 (1979).

11. Wilton, J. R., On ripples, Phil. Mag. 29:688-700 (1915).

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